3a. Sequences: definitions and limits

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Introduction

We partially move back to classical discrete mathematics: sequences (in particular, number sequences) are function of a discrete domain, although technically they can assume any real values.
However, not all sequences have to be number sequences, they can have any values (e.g. a sequence of results of subsequent steps of a given algorithm). We will get back to this case at the end of this presentation.

Number sequences

Number sequence

A *number sequence* is a real-valued function such that its domain is $\mathbb{N}$ (or, at least, an infinite subset of $\mathbb{N}$).

Finite number sequence

A *finite number sequence* is a real-valued function such that its domain is a finite subset of $\mathbb{N}$.

The arguments of a sequence are often termed indices, and its values are termed elements. For a sequence a, we often use the notation a_n instead of $a(n)$.

A note on number sequences

I am not going to remind you definitions of arithmetic and geometric sequences from school, but you should know them. We are going to get back to similarly structured sequences in a section about recurrence relations.

As for the limits of sequences, we can define them in the same way as for the other functions. However, there is no point in computing a limit of a sequence as its arguments approach any number in $\mathbb{R}$: either the sequence is not defined in a neighborhood of such a number or this number belongs to a domain of the sequence, thus the value of the sequence in this point is the same as this limit. Thus, the only non-trivial limit of a sequence may be computed as its arguments approach infinity. For that case, we have an alternative (equivalent) definition of a limit.

Limit of a sequence

Limit of a sequence

A limit of a sequence $(a_n)_{n\in\mathbb{N}}$ is a number $g \in \mathbb{R}$ such that

$$\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0}\ |a_n - g| < \varepsilon.$$

A limit of a sequence $(a_n)_{n\in\mathbb{N}}$ is $+\infty$ if and only if

$$\forall_{M\in\mathbb{R}}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0}\ a_n > M.$$

A limit of a sequence $(a_n)_{n\in\mathbb{N}}$ is $-\infty$ if and only if

$$\forall_{M\in\mathbb{R}}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0}\ a_n < M.$$

Limit of a sequence

If a number sequence has a limit that is a real number, then the sequence is termed *convergent*. Otherwise, it is *divergent*.
Why do we need a second definition of a limit just for sequences if we already have a more general definition for a function and each sequence is a function? Sometimes, the sequence definition is just more convenient to use (in particular, to prove that some functions do not have a limit).

Heine-Borel Theorem

A relation between limits of functions and sequences is the following:

Heine-Borel Theorem

Let $x_0 \in \mathbb{R}$ and let a function f be defined on a certain neighborhood of x_0 (potentially, without the point x_0) . $\lim_{x \longrightarrow x_0} f(x) = g$ if and only if:

$$\forall_{(x_n)_{n\in\mathbb{N}}} \left\{ \left(\forall_{n\in\mathbb{N}}\ x_n \in D_f \wedge \lim_{n\longrightarrow\infty} x_n = x_0 \right) \Rightarrow \lim_{n\longrightarrow\infty} f(x_n) = g \right\}.$$

Heine-Borel Theorem says that computing limits of functions can be transformed into computing limits of sequences (and the other way around).

Heine-Borel Theorem - infinity versions

Heine-Borel Theorem

Let a function f be defined on the interval $(M, +\infty)$, $M \in \mathbb{R}$. Then, $\lim_{x \to \infty} f(x) = g$ if and only if:

$$\forall_{(x_n)_{n\in\mathbb{N}}} \left\{ (\forall_{n\in\mathbb{N}}\ x_n \in D_f \wedge \lim_{n\to\infty} x_n = +\infty) \Rightarrow \lim_{n\to\infty} f(x_n) = g \right\}.$$

Let a function f be defined on the interval $(-\infty, M)$, $M \in \mathbb{R}$. Then, $\lim_{x \to -\infty} f(x) = g$ if and only if:

$$\forall_{(x_n)_{n\in\mathbb{N}}} \left\{ (\forall_{n\in\mathbb{N}}\ x_n \in D_f \wedge \lim_{n\to\infty} x_n = -\infty) \Rightarrow \lim_{n\to\infty} f(x_n) = g \right\}.$$

Heine-Borel Theorem - example

Task

Prove that $\lim_{x \to \infty} \sin x$ does not exist.

By the Heine-Borel theorem, it is sufficient to construct two sequences: (a_n) i (b_n) tending to $+\infty$ as n approaches $+\infty$ such that $\lim_{n \to \infty} \sin a_n \neq \lim_{n \to \infty} \sin b_n$. We define $a_n = n\pi$: it obviosly tends to $+\infty$. In the same time, $\lim_{n \to \infty} \sin a_n = \lim_{n \to \infty} 0 = 0$. On the other hand, if we defined $b_n = 2n\pi + \frac{\pi}{2}$, then b_n tends to $+\infty$ and $\lim_{n \to \infty} \sin b_n = \lim_{n \to \infty} 1 = 1$. $0 \neq 1$, thus $\lim_{x \to \infty} \sin x$ does not exist.

Heine-Borel Theorem - corollary

In particular, from the Heine-Borel Theorem we get to know that we do not need any special techniques for computing limits of sequences other than ones we already know for functions.

Corollary of Heine-Borel Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a number sequence, and f be a real-valued function such that $f(n) = a_n$ for $n \in \mathbb{N}$. If $\lim_{x\to\infty} f(x) = g$, then $\lim_{n\to\infty} a_n = g$.

This conclusion implies that we may simply substitute a real variable (let's say - x) in place of a natural variable n and compute a limit of a function $f(x)$ arising from this substitution when x approaches infinity instead of a limit of a sequence a_n. If the limit of the function f exists, it is the same as the limit of the sequence a_n.

Heine-Borel Theorem - example

Changing sequences into functions allows us to use some tools of calculus: in particular, we can apply the de L'Hospital's rule.

Example

Calculate $\lim\limits_{n\to\infty} \frac{\ln n}{n}$.

By the Heine-Borel Theorem $\lim\limits_{n\to\infty} \frac{\ln n}{n} = \lim\limits_{x\to\infty} \frac{\ln x}{x}$ where $x \in \mathbb{R}$ as long as the former of these limits exists. Then:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] \overset{H}{=} \lim_{x\to\infty} \frac{1}{x} = 0.$$

Heine-Borel Theorem - counterexample

Corollary of Heine-Borel Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a number sequence, and f be a real-valued function such that $f(n) = a_n$ for $n \in \mathbb{N}$. If $\lim_{x\to\infty} f(x) = g$, then $\lim_{n\to\infty} a_n = g$.

Finally, we should note that this conclusion is only an implication, not an equivalence. The fact that a limit of a single sequence exist does not infer existence of a limit of a relevant function. For example, considering the sequence $a_n = \sin(n\pi)$, we easily notice that all its elements (and thus its limit) is 0. However, $\lim_{x\to\infty} \sin(x\pi)$ does not exist.

General sequences

Sequence

Generally, a *sequence* is any function such that its domain is a subset of $\mathbb{N}$. If its domain is additionally finite, we term it a *finite sequence*.

Generally, a sequence does not have to be a number sequence: there can be a sequence of letters or other symbols (so called (*string*)), a sequence of words, graphs, functions etc. . In computer science, a finite sequence is often called a *list* and an infinite (at least, potentially) sequence is called a (*stream*).