1a. Introduction to mathematical logic

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Mathematical logic

Mathematical logic:

- An area of mathematics focused on rules of mathematical reasoning.
- Its aim is to distinguish between correct and incorrect reasoning and to provide proper language for formulating mathematical theorems and definitions.
- Logic constitutes a foundation of modern computer science. In particular, it is widely used in construction and analysis of programming languages.
- The laws of mathematical logic are represented in computer hardware by logic gates and circuits.

Plan of this section

- Terminology and notation (in particular, truth tables);
- Propositional calculus;
- Types of proofs and counterexamples;
- Elements of set theory;
- Introduction to logic circuits.

Sentence

Sentences in logic

A *sentence* (or a *proposition*) is a statement to which we can assign exactly one *truth value*: either *true* or *false*.

In propositional calculus, we denote sentences by small letters: usually p, q, r, ...

Examples

Exemples of sentences:

- Napoleon Bonaparte is a president of Poland. (false)
- $2 + 2 = 4$. (true)
- 4 is positive and 3 is negative. (compound sentence, false)
- Each even integer larger than 2 is a sum of two prime numbers. (Goldbach's Conjucture, it is either true or false, we just do not know so far).

Counterexamples

The expressions below are not sentences (in mathematical logic):

- Is it raining? (question, no truth value)
- No more war! (phrase, no truth value until it is transformed into a sentence)
- Study! (command, no truth value)
- If you don't stop shouting, I'll leave. (this might be a sentence; it depends on how much we know about the intentions of the speaker. Without extra information, it is hard to determine the truth value of this statement).
- $x - y = y - x$. (y and x are not defined, so we do not know the truth value. This is a *predicate*.)

Unclear sentences

It is possible that a statement is technically a sentence, but its truth value is difficult to recognize (usually because of lack of precision).

- Students are rich.
- It is hot today.
- Mathematics is interesting.
- For each A such that $A^2 = 0$, it holds that $A = 0$. (We do not know where A comes from. If we deal with real numbers, the sentence is true. If we deal with real-valued matrices, this sentence is false.)

We try to avoid such sentences or try to make them unambiguous.

Predicates

Sometimes in the context of mathematics or programming, we use statements which contain a free variable and only after obtaining extra information on that variable we can determine the truth value of the statement. We term them **predicates**.

- $x > 0$. (predicate of a variable x)
- A knows B. (predicate of variables A and B)
- n mathematicians know the number k which satisfies the sentence p. (predicate of variables n, k, p)

In propositional calculus, we denote predicates in the same way as functions e.g. $p(x)$, $q(A, B)$, $\varphi(n, k, p)$.

Predicates and quantifiers

Theorems and definitions are often presented as predicates transformed into a sentence by one of certain phrases known as quantifiers. The two most important (and widely used) quantifiers are:

- The **universal quantifier** standing for the phrase "for all". We denote it by $\forall$. For example, $\forall_{x\in\mathbb{R}}\ x^2 \geqslant 0$. Sentences of type $\forall_{x\in X} p(x)$ are true if and only if after substituting any x from the set X the sentence $p(x)$ is true.
- The **existential quantifier** standing for the phrase "there exists". We denote it by $\exists$. For example, $\exists_{x\in\mathbb{R}}\ x < 0$. Sentences of type $\exists_{x\in X} p(x)$ are true if and only if there exists at least one x from the set X such that $p(x)$ is true.

There exist other quantifiers (e.g. $\exists!$ means "there exists exactly one") but they fall beyond the scope of this course.

Examples

- X passed the exam. (Predicate of variable X)
- For every X from the set of first-year students ($\forall_{X \in S}$, S - the set of first-year students), X passed the exam. (a sentence, probably false)
- There exists X from the set of first-year students ($\exists_{X \in S}$, S - the set of first-year students) such that X passed the exam. (a sentence, probably true)
- $\forall_{n \in \mathbb{Z}, n>1} \exists_{p_1, p_2 \in \mathcal{P}} \; 2n = p_1 + p_2$. (Goldbach's Conjecture, shortened)

We sometimes do not write quantifiers if their existence is obvious. For example, the commutativity property of addition of real numbers we usually write down as $x + y = y + x$, though formally it should be: $\forall_{x,y \in \mathbb{R}} \; x + y = y + x$.

Proofs and counterexamples

General rule: to determine if something is true (or false) you should present a rigorous proof. Very often theorems take the form: $\forall_{x \in X} p(x)$. Then:

- If $\forall_{x \in X} p(x)$ is supposed to be true, we need to prove $p(x)$ for each $x \in X$. It is not enough to confirm it for a few (or a few milliards) of examples. Goldbach's Conjucture is confirmed for numbers up to 10^{17} but it is not sufficient to consider that it is proven.
- However, if we want to falsify such a theorem, we need to show that just one sentence $p(x)$ is false. Such a "falsifying sentence" is called a **counterexample**.
- A sentence: *Each natural number is smaller than million* ($\forall_{n \in \mathbb{N}}\ n < 10^6$) is true for first million cases. However, the number $n = 10^9 + 1$ is a counterexample, so the entire sentence is false.

Truth functions

Determining if a simple sentence is true or false usually falls beyond the scope of logic. In logic, we usually try to determine the truth value of compound sentences assuming that the truth value of their simple components is known. This is the focus of so called propositional calculus.

We construct compound sentences out of simple base sentences by using **truth functions**, namely *actions* defined on the set of sentences and interpreted as logical connectives between base sentences. The most popular and important truth functions (from which any other truth function can be constructed) are: negation (*It is not true that...*), conjunction (*and*), alternative (*or*), implication (*If..., then*), equivalence(*...if and only if...*).

Henceforth, in our logical computations 1 stands for true and 0 stands for false.

Truth tables

Truth tables (or *truth matrices*) constitute a basic tool of propositional calculus. We will use them both to define the truth functions and to determine the truth value of compound sentences.

Truth table

A truth table of a compound sentence that consists of simple sentences is a table which provides truth values of this compound sentence for each combination of truth values of component sentences.

Negation

A negation is the only truth function that operates only on one prepositional variable. For any sentence p, a sentence "It is not true that p" or in short "not-p" is a **negation** of p, denoted by $\neg p$ or $\sim p$. It is true only if the base sentence p is false as represented by the table:

p	$\sim p$
1	0
0	1

For example, a sentence "2+2=4" is true, thus "It is not true that 2+2=4" or, in short, "$2 + 2 \neq 4$" must be false.

Conjunction

A **conjunction** of base sentences p and q is a sentence "p and q" (namely, injecting a connective "and" between them) which is denoted by $p \wedge q$. A conjunction is true if and only if both base sentences are true, as represented by the table:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

For example "2+2=5 and it is raining in Krakow now" is always false (whether it is raining or not) because the first component of the sentence is false.

Alternative

An **alternative** of base sentences p and q is a sentence "p or q" (namely, injecting a connective "or" between them) which is denoted by $p \vee q$. An alternative is true if and only if at least one of the base sentences is true, as represented by the table:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For example, a sentence „2+2=4 or it is raining in Krakow now" is always true (whether it is raining or not) because the first component of the sentence is true.

Curiosity: There exists a truth function called an exclusive alternative (XOR, exclusive or, legalese or) which is true if and only if exactly one of the base sentences is true.

Implication

An **implication** of base sentences p and q is a sentence "p implies q"/"If p, then q" which is denoted by $p \Rightarrow q$. An implication is true if and only if p is false or q is true, as represented by the table:

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Arguably, the implication is the most important truth function for mathematicians, because most theorems are in the form of implications (e.g. „If a, b, c are the lengths of sizes of the right triangle (c -hypothenuse), then $a^2 + b^2 = c^2$") and their proofs can be presented as sequences of implications.

Implication

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Implication

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

For many, the implication table is counterintuitive: *why any false sentence implies any true sentence and any false implies any false*? A standard explanation is as follows: we do not discuss if any of the component sentences (p, q) is true but if the reasoning process is correct. Starting from an incorrect premise, a correct reasoning can lead us anywhere: to either true or false conclusion.

Implication: example

For example, let the premise be $1 = 2$ (obviously false). By the laws of arithmetics, we can multiply both sides of a true equality by the same real number and the resulting equality is still true.
If we multiply both sides of the premise by 2, we obtain $2 = 4$. As the reasoning is correct, the implication $1 = 2 \Rightarrow 2 = 4$ is true while both base sentences are false.
Alternatively, if we multiply both sides of the premise by 0, we obtain $0 = 0$, which is trivially true. The reasoning is again correct, so the implication $1 = 2 \Rightarrow 0 = 0$ is true, even though the premise is false and the conclusion is true.

Equivalence

An **equivalence** of base sentences p and q is a sentence "p if and only if (in short: iff) q"/"p is equivalent to q" which is denoted by $p \Leftrightarrow q$. An equivalence is true if and only if both base sentences have the same truth value, as represented by the table:

p	q	$p \Leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Many theorems are formulated as equivalences. In fact, an equivalence may be perceived as "a two-way implication".

$$(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)].$$

Analysis of truth values for sentences

We can apply our understanding of basic truth functions to determine truth values of compound sentences. For example, assume that we need to evaluate what is the truth value of the following sentence for each combination of truth values of component sentences.

Task

If Aurora plays chess well, then she is wise, and if she is not wise or speaks Swahili, then she does not play chess well.

First, we need to identify base component sentences (they cannot be decomposed into simpler sentences; their truth values cannot be determined only by means of logic). There are three such sentences "Aurora plays chess well" (p), "Aurora is wise" (q) and "Aurora speaks Swahili"(r).

Logical form of a sentence

Task

If Aurora plays chess well, then she is wise, and if she is not wise or speaks Swahili, then she does not play chess well.

Applying the notation $p =$"Aurora plays chess well", $q =$"Aurora is wise", $r =$"Aurora speaks Swahili" we present the sentence above in the *logical form*:

$$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)] .$$

We will use the truth table to determine the truth values of this sentence.

Truth table

$$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)] .$$

We list all the possible combinations of truth values of base sentences (for n sentences, there are 2^n combinations).

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Truth table

$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)]$.

The main connective (namely, the last applied function) is $\wedge$, so we begin by determining the truth value of the sentence on the left of it.

p	q	r	$p \Rightarrow q$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Truth table

$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)]$.

Next, we determine truth values for the simplest compound sentences on the right side (for simplicity, we denote $p \Rightarrow q$ as L for the left side of the main sentence):

p	q	r	L	$(\sim q)$	$(\sim p)$	$(\sim q) \vee r$
1	1	1	1	0	0	1
1	1	0	1	0	0	0
1	0	1	0	1	0	1
1	0	0	0	1	0	1
0	1	1	1	0	1	1
0	1	0	1	0	1	0
0	0	1	1	1	1	1
0	0	0	1	1	1	1

Truth table

$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)]$.

We are now ready to determine the values of the sentence on the right of $\wedge$, namely $((\sim q) \vee r) \Rightarrow (\sim p)$ (denoted by R).

p	q	r	L	$(\sim q)$	$(\sim p)$	$(\sim q) \vee r$	R
1	1	1	1	0	0	1	0
1	1	0	1	0	0	0	1
1	0	1	0	1	0	1	0
1	0	0	0	1	0	1	0
0	1	1	1	0	1	1	1
0	1	0	1	0	1	0	1
0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1

Truth table

$(p \Rightarrow q) \wedge [((\sim q) \vee r) \Rightarrow (\sim p)]$.

In the last column, applying columns L, R, and the definition of conjunction, we obtain the truth values for the entire sentence.

p	q	r	L	$(\sim q)$	$(\sim p)$	$(\sim q) \vee r$	R	$L \wedge R$
1	1	1	1	0	0	1	0	0
1	1	0	1	0	0	0	1	1
1	0	1	0	1	0	1	0	0
1	0	0	0	1	0	1	0	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	0	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

Truth table and truth values

p	q	r	L	$(\sim q)$	$(\sim p)$	$(\sim q) \vee r$	R	$L \wedge R$
1	1	1	1	0	0	1	0	0
1	1	0	1	0	0	0	1	1
1	0	1	0	1	0	1	0	0
1	0	0	0	1	0	1	0	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	0	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

The table informs us that the entire sentence is false only in three cases: when all 3 base sentences are true, when only the sentences p and q are true, and when only the sentence p is true.

Truth values of a sentence

The table informs us that the entire sentence is false only in three cases: when all 3 base sentences are true, when only the sentences p and q are true, and when only the sentence p is true.
Recalling our notation: $p =$"Aurora plays chess well"', $q =$"Aurora is wise" $r =$"Aurora speaks Swahili", we obtain that the original sentence is false if Aurora plays chess well and speaks Swahili (no matter if she is wise or not) or Aurora plays chess well but neither speaks Swahili nor is wise. Otherwise, the original sentence is true.
The rest of analysis does not depend on logic: we need to determine the truth values of basic sentences by other means.

Tautology and contradiction

The sentence that we just studied had different truth values for different values of components (was *logically contingent*). This is not always the case.

Tautology

A **tautology** is a compound sentence which is true under any choice of truth values for its component sentences.

Tautology is characterized by a truth table which has only 1s in its results column.

Contradiction

A **contradiction** is a compound sentence which is false under any choice of truth values for its component sentences.

Naturally, p is a tautology if and only if $\sim p$ is a contradiction.

Tautology and contradiction - examples

Simple tautologies, known as the laws of logic, are particularly useful to conduct reasoning to prove (or disprove) more complicated claims. One of such tautologies is $p \vee (\sim p)$. We can prove it by constructing its truth table:

p	$\sim p$	$p \vee (\sim p)$
1	0	1
0	1	1

A classical contradiction is $p \wedge (\sim p)$ (an exercise for you).

Basic laws of logic

We are now going to study some basic laws of logic, namely tautologies that are most useful for constructing proofs and studying reasoning. Remember that this is not an exhaustive list, we are going to discuss only the most popular ones!
In the list of laws, we denote any true sentence by 1 and any false sentence by 0.

Laws of simplification and idempotence

$p \vee p \Leftrightarrow p$; $p \wedge p \Leftrightarrow p$;
$p \vee 1 \Leftrightarrow 1$; $p \vee 0 \Leftrightarrow p$; $p \wedge 0 \Leftrightarrow 0$; $p \wedge 1 \Leftrightarrow p$.

Laws of commutation and association

Laws of commutation and association

$(p \vee q) \Leftrightarrow (q \vee p)$; $(p \wedge q) \Leftrightarrow (q \wedge p)$; $(p \Leftrightarrow q) \Leftrightarrow (q \Leftrightarrow p)$;
$[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$; $[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$.

Commutation laws state that the ordering of sentences in conjunctions, alternatives and equivalences does not matter (it matters for an implication). Association laws state that sequences consisting exclusively of conjunctions or exclusively from alternatives can be read in any order (so there is no need to add parentheses to them).

Law of contraposition and other transformations of implications

Law of contraposition

$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.

$(\sim q \Rightarrow \sim p)$ is known as a **contraposition** or a **transposition** of $(p \Rightarrow q)$ and has the same truth value. This is used in so called "proofs by contradiction" or "proofs by contraposition".

A sentence $q \Rightarrow p$ is termed a **converse** to $(p \Rightarrow q)$ and, as well as its contraposition named an **inverse** $(\sim p \Rightarrow \sim q)$ generally has different truth values than the original sentence.

Contraposition and converse - examples

Let us consider the implication: "If it is raining, then it is cloudy". A contraposition (equally true) of this implication is a sentence: "If it is not cloudy, then it is not raining".
A converse of the original implication would be "If it is cloudy, then it is raining", and an inverse: "If it is not raining, it is not cloudy". The latter two sentences are not equivalent to the two former ones.

Reductio ad absurdum

Reductio ad absurdum

$(p \Rightarrow q) \Leftrightarrow [(p \wedge \sim q) \Rightarrow 0]$.

This is the basis of an important technique of proving theorems by "leading to contradiction". Instead of conducting direct proof (going from promises to a conclusion), one considers what would happen if the premises were true and the conclusion was false and shows that (we will get back to this technique later) it would imply a claim that is obviously false.

Negation laws

Double negation

$\sim (\sim p) \Leftrightarrow p.$

De Morgan's Theorems

$\sim (p \wedge q) \Leftrightarrow (\sim p \vee \sim q);$
$\sim (p \vee q) \Leftrightarrow (\sim p \wedge \sim q).$

De Morgan's Theorems state that if we negate either a conjunction or an altenative, then we need not only to negate the compontent sentences but also replace a $\vee$ connectives with $\wedge$ ones (and the other way around).

Negation laws

Negation of implication

$\sim (p \Rightarrow q) \Leftrightarrow (p \wedge (\sim q)).$

The negation of equivalence is more complex, and thus it is not a basic law of logic, but we can easily construct it by applying other laws. This is an example of a classical propositional calculus (transforming sentences into equivalent statements applying laws of logic): We know that
$(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)].$

Propositional calculus - example

$(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$.
Thus, to negate an equivalence, we need to negate a conjunction of two implications. By applying already mentioned negation laws we obtain:

$$\sim [(p \Rightarrow q) \wedge (q \Rightarrow p)] \Leftrightarrow [\sim (p \Rightarrow q) \vee \sim (q \Rightarrow p)] \Leftrightarrow$$

$$\Leftrightarrow [(p \wedge (\sim q)) \vee (q \wedge (\sim p))].$$

Negation of equivalence

Thus, we obtained an extra general "law":

$$\sim (p \Leftrightarrow q) \Leftrightarrow [(p \wedge (\sim q)) \vee (q \wedge (\sim p))].$$

In the "proof" above we sneakily used probably the most important law of logic.

Transitivity of implications and equivalences

Laws of transitivity

$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$;
$[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$.

These laws above are also known as rules of inference or chain rules and form the basis of all direct mathematical proofs. Such proofs are analogous to the sequences of equalities you know from arithmetics: the premise implies fact 1, fact 1 implies fact 2, fact 2 implies fact 3... and finally we have a sequence of implications that (hopefully) ends in a conclusion: $p \Rightarrow f_1 \Rightarrow f_2 \ldots \Rightarrow f_n \Rightarrow c$ (these implications can be replaced by equivalences) Therefore, standard proofs are correct only because of laws of transitivity.

Commutation of quantifiers

Commutation of quantifiers

$\forall_x \forall_y \varphi(x, y) \Leftrightarrow \forall_y \forall_x \varphi(x, y)$;
$\exists_x \exists_y \varphi(x, y) \Leftrightarrow \exists_y \exists_x \varphi(x, y)$;
$\exists_y \forall_x \varphi(x, y) \Rightarrow \forall_x \exists_y \varphi(x, y)$.

These laws state that the ordering of the same (i.e. all universal or all existential) quantifiers does not matter. However, the ordering of different quantifiers is important (there is only an implication in the third line).

Commutation of quantifiers

Partial commutation of quantifiers

$\exists_y \forall_x \varphi(x, y) \Rightarrow \forall_x \exists_y \varphi(x, y)$.

Illustration of the last law: Let $\varphi(x, y)$ denote: "A pair of shoes y fits person x". Then, naturally, it is more plausible that for each person there exists a fitting pair of shoes than that one pair of shoes fits every person: therefore these two statements are not equivalent. More precisely, if one, universal pair of shoes is comfortable for everyone, then each person can find a pair of shoes fitting them (e.g. the universal one). At the same time, we can assume that although it is probably possible that each person has a comfortable pair of shoes, while a pair of boots good for everyone probably does not exist.

Negation of quantifiers

Negation of quantifiers

$\sim \forall_x \varphi(x) \Leftrightarrow \exists_x \sim \varphi(x)$;
$\sim \exists_x \varphi(x) \Leftrightarrow \forall_x \sim \varphi(x)$;

If we negate a sentence containing a quantifier, then we not only need to negate the predicate under the quantifier but change the quantifier itself: from the universal to existential or the other way around.

Negation of quantifiers

Negation of quantifiers

$\sim \forall_x \varphi(x) \Leftrightarrow \exists_x \sim \varphi(x)$;
$\sim \exists_x \varphi(x) \Leftrightarrow \forall_x \sim \varphi(x)$;

For example, if we try to disprove the claim "All cows are black" we can just point out one non-black cow proving equivalently that "There exists a cow which is not black". However, to disprove the claim "There exists an integer divisible by 4 which is not even", it is not sufficient to point out one counterexample; we need to prove the equivalent sentence "Every integer divisible by 4 is even".

Negation of quantifiers - example

Task

Negate the sentence:

$$\exists_{x\in\mathbb{R}}\forall_{y\in\mathbb{N}}\exists_{\alpha>0}\ \ (2x^2+y^2=\alpha\Rightarrow x>y+\alpha).$$

First, we need to change all quantifiers to "opposite": $\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{N}}\forall_{\alpha>0}$. Secondly, we need to negate a predicate under quantifiers (an implication). To sum up, we obtain:

$$\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{N}}\forall_{\alpha>0}(2x^2+y^2=\alpha\wedge x\leqslant y+\alpha).$$