Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy" jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

Numerical methods

Handouts for students

7. Numerical integration

przygotowane w ramach projektu „Uruchomienie go kierunku studiów Informatyka Stosowana odpowiedzią na wanie rynku pracy" ze środków Programu Operacyjnego udzki współfinansowanego ze środków Europejskiego Społecznego nr umowy UDA – POKL.04.01.01-00-011/09-

I. Introductory requirements

It is required to know the concepts of:

- indefinite integral;
- definite integral;

and be able to:

- calculate basic indefinite and definite integrals;
- calculate differentials of functions;
- find lower and upper bounds of a function defined in an interval.

II. Classes

Task 1. Calculate the following integrals analytically and using the quadrature formulae (use $n = 1$).

a) $\int_{-1}^{5}(x+1)dx$; b) $\int_{-2}^{4}(x^3-3)\ dx$; c) $\int_{-3}^{3}(2x-x^4)\ dx$.

Task 2. Calculate the following integrals analytically and using the quadrature formulae with $n = 1$ and $n = 2$. Compare the results and the actual errors with theoretical estimates of errors.

a) $\int_0^{\pi} \sin^2 x\ dx$,

b) $\int_{-2}^{4} \frac{x^2-2}{x+3} dx$.

Task 3. Using Simpson 1/3 formula or Simpson 3/8 formula evaluate three approximations of the value:

a) $\ln 2$,

b) $\arctan 3$.

III. Homework

Task 1. Use the quadrature formulae for the following integrals. Consider $n = 1$ and $n = 2$.

a) $\int_1^4 \left(x^2 - x - \frac{1}{x}\right) dx$, b) $\int_0^{\pi} x^2 \cos^2 x \; dx$, c) $\int_{-5}^{1} \frac{x}{x-2} dx$.

Task 2. Evaluate the number of subintervals of the interval $[1, 2]$ that should be used in order to get the value of the following integral with an error not greater than:

a) 10^{-5}, b) 10^{-7}, c) 10^{-9}.

$$\int_1^2 \frac{1}{x} dx$$

Task 3. The risk of a head injury in a car accident can be evaluated using a HIC indicator (Head Injury Criterion). It is defined by a formula:

$$HIC = \max_{0 \le t_1 \le t_2 \le T} (t_2 - t_1) \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a(t)\, dt \right)^{2,5}$$

where $a(t)$ is a function that describes the inertia forces which induce the injury (it is proportional to acceleration), and T is the braking time of the car moving initially with the velocity of 50 km/h. Estimate the value of this index by performing the following steps:

- evaluate the mean value of the function $a(t)$ on the interval $[50, 110]$

$$a(t) = \frac{16\,400}{(t-68)^2 + 400} + \frac{1\,480}{(t-93)^2 + 18}$$

  calculating integral of this function using Simpson 1/3 formula or Simpson 3/8 formula with $n = 2$ and dividing it by the length of this interval, i.e.

$$\bar{a} = \frac{1}{60} \int_{50}^{110} a(t)\, dt$$

  (argument t of function $a(t)$ is in milliseconds);
- evaluate approximated value of HIC indicator calculating:

$$HIC \approx (\bar{a})^{2,5} \cdot 0{,}06$$

  where $0{,}06$ is the length of the window of integration in seconds;
- value of HIC above $1\,000$ indicate threat to a human life. Compare the above result with the value of the HIC indicator for a function $a_A(t)$, that describes the case of an airbag in the car:

$$a_A(t) = \frac{22\,000}{(t-74)^2 + 500}$$

IV. Answers

Task 1.

a) $\int_1^4 \left(x^2 - x - \frac{1}{x}\right) dx = \frac{27}{2} - 2\ln 2 \approx 12{,}1137$

Formula	Trapezoid	Simpson 1/3	Simpson 3/8
$n = 1$	$\frac{129}{8} = 16{,}125$	$\frac{483}{40} = 12{,}075$	$\frac{387}{32} = 12{,}094$
$n = 2$	$\frac{1\,047}{80} = 13{,}088$	$\frac{88\,149}{7\,280} = 12{,}108$	$\frac{27\,129}{2\,240} = 12{,}111$

b) $\int_0^{\pi} x^2 \cos^2 x \; dx = \frac{\pi^3}{6} + \frac{\pi}{4} \approx 5{,}953111$

Formula	Trapezoid	Simpson 1/3	Simpson 3/8
$n = 1$	$\frac{\pi^3}{2} \approx 15{,}503$	$\frac{\pi^3}{6} \approx 5{,}168$	$\frac{17\pi^3}{96} \approx 5{,}490$
$n = 2$	$\frac{\pi^3}{4} \approx 7{,}752$	$\frac{3\pi^3}{16} \approx 5{,}814$	$\frac{73\pi^3}{384} \approx 5{,}894$

c) $\int_{-5}^{1} \frac{x}{x-2} dx = 6 - 2\ln 7 \approx 2{,}10818$

Formula	Trapezoid	Simpson 1/3	Simpson 3/8
$n = 1$	$-\frac{6}{7} \approx -0{,}857$	$\frac{12}{7} \approx 1{,}714$	$\frac{66}{35} \approx 1{,}886$
$n = 2$	$\frac{15}{14} \approx 1{,}071$	$\frac{1\,563}{770} \approx 2{,}03$	$\frac{579}{280} \approx 2{,}068$

Task 2.

Estimate of error	Trapezoid formula	Simpson 1/3 form.	Simpson 3/8 form.
10^{-5}	$n = 130$	$n = 6$	$n = 5$
10^{-7}	$n = 1\,291$	$n = 17$	$n = 14$
10^{-9}	$n = 12\,910$	$n = 54$	$n = 44$

Task 3.

Formula	Simpson 1/3	Simpson 3/8
Without an airbag	571	688
With an airbag	297	297