KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI
UNIWERSYTET EKONOMICZNY W KRAKOWIE
EDUKACJA DLA PRZEDSIĘBIORCZOŚCI
UNIA EUROPEJSKA
EUROPEJSKI FUNDUSZ SPOŁECZNY

Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy" jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

Numerical Methods

Handouts for students

5. Iterative methods for nonlinear equations

informatyka stosowana

przygotowane w ramach projektu „Uruchomienie
go kierunku studiów Informatyka Stosowana odpowiedzią na
wanie rynku pracy" ze środków Programu Operacyjnego
udzki współfinansowanego ze środków Europejskiego
Społecznego nr umowy UDA – POKL.04.01.01-00-011/09-

UNIWERSYTET EKONOMICZNY W KRAKOWIE

I. Introductory requirements

It is required to know the concepts of:

- secant, tangent line;
- derivative of function of one variable;
- characterisation of properties of function via derivatives;
- jacobian;
- inverse matrix;

and be able to:

- solve linear equations;
- given two points, find the formula of the line;
- compute first and second derivatives of function of one variable and multiple variables;
- find tangent line to the graph of function.

II. Classes

Task 1. Bound the roots of given equations to the intervals of length 1:

a) $x^2 - 5 = 0$;
b) $x^3 + 6x^2 - 15x - 1 = 0$;
c) $4x^3 - 2x^2 + 3 = 0$.

Task 2. For the equations and intervals from the task 1, find the second approximation of each solution using the bisection method. Evaluate the absolute errors of these approximations. How many iterations of the method is needed to ensure that the error of the approximation in no greater than 0,001?

Task 3. Determine if the mapping $g(x) = \frac{1}{2}x(x-1)$ satisfies the assumptions of the theorem on iteration method convergence:

a) on the interval $[0,1]$;
b) on the interval $\left[-\frac{1}{4}, \frac{3}{4}\right]$.

Task 4. Determine which of the following reformulations of the equation $x^2 - 5 = 0$ guarantees the convergence of the Iteration Method. Use it to find the second approximation of the solution:

a) $x^{(n+1)} = 5 + x^{(n)} - \left(x^{(n)}\right)^2$
b) $x^{(n+1)} = \frac{5}{x^{(n)}}$
c) $x^{(n+1)} = 1 + x^{(n)} - \frac{1}{5}\left(x^{(n)}\right)^2$
d) $x^{(n+1)} = \frac{1}{2}\left(x^{(n)} + \frac{5}{x^{(n)}}\right)$.

For each possible case, evaluate the absolute error of the second approximation (use the appropriate theorem from the lecture).

Task 5. Evaluate the first two iterations in the Newton's Method for the polynomial $p(x) = 4x^3 - 2x^2 + 3$ in the interval $(-1,0)$. Prove that there exists exactly one zero of that polynomial in this interval. Justify the convergence of the method and calculate the absolute error of the second approximation of the solution on the basis of the appropriate theorem from the lecture.

Task 6. Using the Regula Falsi find the first two approximations of the positive root of the equation

$$x^3 + x^2 - 3x - 3 = 0.$$

Justify the convergence of the method and calculate the absolute error of the second approximation of the solution on the basis of the appropriate theorem from the lecture.

Task 7. Find the third approximation of the number $\sqrt[3]{2}$ using:

a) the Bisection Method (starting from the interval of the length 1);
b) the Iteration Method;
c) the Secant Method;
d) the Tangent Method.

Task 8. The functions of supply and demand for bananas are expressed by formulas $P(x) = \frac{3}{2(x+1)}$ and $Q(x) = e^x - 1$ respectively, where x represents the price of bananas. Using the Iteration Method calculate economic equilibrium price for bananas, taking as an answer the second iteration of this method.

Task 9. The dependence between the specific heat of water c at a temperature t, and the temperature is described by formula $c = 2t + e^t(1 - t)$. Calculate the temperature such that the specific heat of water c reaches the greatest value and calculate this value. Take as an answer the first iteration of the Newton's Method.

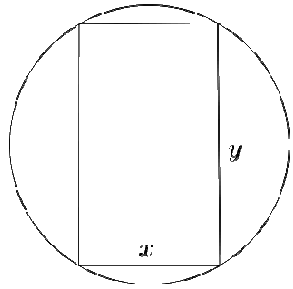

Fig. 1

Task 10. The section modulus of a rectangular beam, lying horizontally, is given by formula , where is the width, – the height of the beam section. How to cut a cylindrically shaped stem, whose base has a diameter equal 2, to obtain a rectangular beam with the highest section modulus (see Fig. 1)? As an answer take the second iteration of the Secant Method.

Task 11. Using the Newton's Method evaluate the given number of iterations:

a) starting point , one iteration

b) starting point , two iterations

Task 12. Jaś Kowalski is an applied computer science student at the Cracow University of Economics. He went on vacation to her grandparents in France who are famous mathematicians. Since he has not seen his grandparents for a long time, he asked them how old they were. The old men answered him as follows: "Think about the numbers describing how old we are. The difference of the squares of these numbers is 240, and the difference of cubes of these numbers is 21602 ". Jaś was also asked how old he thinks his grandparents are. Jaś gave them this answer: "You are very vital and vivacious. I suppose you are 60 years old but taking into account the social stereotype connected with marriage that a man should be elder than a women, I guess that grandma is 58 years old, and grandpa is 60 years old". Grandpa replied to this as follows: "You're right, but this is not the exact answer". Grandmother asked him: 'Do you know how to calculate how old we are? Or do you need a clue? ". Jaś quickly and proudly replied: "I do not need a clue, I'm going to use Newton's Method for nonlinear equations. I learned this method during the course of numerical methods at the Cracow University of Economics". What is the correct answer that Jaś should give?

III. Homework

Task 1. For the given equations:

1. localize at least one of the solutions and if this is possible, indicate the fixed end of the interval;
2. find the first four aproximations of the solution in the Bisection Method (starting from the interval of the length 1);
3. find the first four aproximations of the solution in the Iteration Method;
4. find the first four aproximations of the solution in the Falsi Method;
5. find the first four aproximations of the solution in the Newton's Method.

a) $x^3 + 3x^2 - 2 = 0$;
b) $\ln x + x + 1 = 0$;
c) $2^{x-3} + x - 2 = 0$.

Task 2. For the equation $x^2 - a = 0$:

a) derive the formula for the $(k+1)$-st approximation in the Newton's Method;
b) taking $a = 2$, $x_0 = 1$ calculate x_2.

Task 3. Evaluate the fifth approximation of $\sqrt[4]{3}$ in:

a) the iteration method;
b) the secant method;
c) the tangent method.

Task 4. A toy airplane, powered by batteries, thrown upward in the air at an angle to the horizontal, moves along path described by formula $h(x) = 3x - x\ln x - \frac{x^2}{2}$. Find the maximum altitude of the airplane. As an answer take the second iteration of the Newton's Method.

Task 5. Find the first two approximations of the solutions to the given systems of nonlinear equations using the Newton's Method:

a) $\begin{cases} x^2 + y^2 + z^2 = 1 \\ 2x^2 + y^2 - 4z = 0, \\ 3x^2 - 4y + z^2 = 0 \end{cases}$ $(x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$;

b) $\begin{cases} x^3 + y^3 - 3xy = 0 \\ x^2 + y^2 = 1 \end{cases}$, $(x_0, y_0) = (1, -1)$.

IV. Answers

Task 1.

a)

1. $\bar{x} \in (0,1)$, $b = 1$ is the fixed end
2. $x_0 = \frac{1}{2}$; $x_1 = \frac{3}{4}$, $x_2 = \frac{5}{8}$; $x_3 = \frac{11}{16}$; $x_4 = \frac{23}{32} = 0{,}71875$.
3. $g(x) = \sqrt{\frac{2}{x+3}}$;
   $x_0 = 0$; $x_1 = 0{,}8164965809$, $x_2 = 0{,}7239066380$; $x_3 = 0{,}7328508664$;
   $x_4 = 0{,}7319723532$
4. $x_0 = 0$; $x_1 = 0{,}5000000000$; $x_2 = 0{,}6800000000$;
   $x_3 = 0{,}7215415460$; $x_4 = 0{,}7299774348$;
5. $x_0 = 1$; $x_1 = 0{,}7777777778$; $x_2 = 0{,}7337566138$;
   $x_3 = 0{,}7320533217$; $x_4 = 0{,}7320508076$;

b)

1. $\bar{x} \in (e^{-2}, 1)$, $a = e^{-2}$ is the fixed end
2. $x_0 = \frac{1}{2}$; $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{8}$; $x_3 = \frac{5}{16}$; $x_4 = \frac{9}{32} = 0{,}28125$.
3. $g(x) = \frac{1}{9}(8x - 1 - \ln x)$;
   $x_0 = 1$; $x_1 = 0,(7)$; $x_2 = 0{,}6081707390$; $x_3 = 0{,}4847406142$;
   $x_4 = 0{,}4002295844$;
4. $x_0 = 1$; $x_1 = 0{,}3963239665$; $x_2 = 0{,}3043158570$;
   $x_3 = 0{,}2845365639$; $x_4 = 0{,}2799144160$;
5. $x_0 = e^{-2}$; $x_1 = 0{,}2384058440$; $x_2 = 0{,}2760175356$;
   $x_3 = 0{,}2784560921$; $x_4 = 0{,}2784645428$.

c)

1. $\bar{x} \in (1,2)$, $b = 2$ is the fixed end
2. $x_0 = \frac{3}{2}$; $x_1 = \frac{7}{4}$, $x_2 = \frac{13}{8}$; $x_3 = \frac{25}{16}$; $x_4 = \frac{51}{32} = 1{,}59375$.
3. $g(x) = 2 - 2^{x-3}$;
   $x_0 = 1$; $x_1 = 1{,}75$; $x_2 = 1{,}579551792$; $x_3 = 1{,}626403772$;
   $x_4 = 1{,}614071960$;
4. $x_0 = 1$; $x_1 = 1{,}6$; $x_2 = 1{,}616175042$; $x_3 = 1{,}616653058$;

$x_4 = 1{,}616667220;$

5. $x_0 = 2; x_1 = 1{,}628687208; x_2 = 1{,}616678203; x_3 = 1{,}616667652;$
$x_4 = 1{,}616667652;$

Task 2.

a) $x_{k+1} = \frac{x_k^2+a}{2x_k}$

b) $x_2 = \frac{17}{12}$

Task 3.

a) $x_5 = 1{,}091652323$ $(g(x) = x - 0.01x^4 + 0.03, x_0 = 1)$

b) $x_5 = 1{,}300879139$ $(\bar{x} \in (1{,}2), x_0 = 1)$

c) $x_5 = 1{,}316074013$ $(\bar{x} \in (1{,}2), x_0 = 2)$

Task 4. $h'(x) = 2 - \ln x - x$. By graphical method we see that this equation has exactly one solution belonging to the interval $(1{,}2)$. Using the Newton's Method we get: $x_0 \coloneqq 1,\ x_1 = \frac{3}{2},\quad x_2 = \frac{9}{5} - \frac{3}{5}\ln\frac{3}{2} \approx 1{,}556721$. Since $h''(x) = -\frac{1}{x} - 1 < 0$ for $x \in (1{,}2)$, then at $x \approx 1{,}556721$ function $h(x)$ reaches its maximum, which approximately equals 2,76949666.

Task 5.

a) $X_1 = \left[\frac{7}{8}, \frac{1}{2}, \frac{3}{8}\right],\ X_2 = \left[\frac{3273}{4144}, \frac{147}{296}, \frac{219}{592}\right];$

b) $X_1 = \left[\frac{1}{2}, -1\right],\ X_2 = \left[\frac{23}{72}, -\frac{139}{144}\right].$