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NARODOWA STRATEGIA SPÓJNOŚCI
UNIWERSYTET EKONOMICZNY W KRAKOWIE
EDUKACJA DLA PRZEDSIĘBIORCZOŚCI
UNIA EUROPEJSKA
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Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy"
jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

Numerical Methods

Handouts for students

4. Eigenvalues and eigenvectors

4.1. Fundamental definitions, properties and theorems
4.2. Localization of eigenvalues
4.3. Power method

informatyka stosowana

przygotowane w ramach projektu „Uruchomienie
go kierunku studiów Informatyka Stosowana odpowiedzią na
wanie rynku pracy" ze środków Programu Operacyjnego
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UNIWERSYTET EKONOMICZNY W KRAKOWIE

I. Introductory requirements

It is required to know the concepts of:

- vector, matrix, determinant of a matrix, trace of a matrix;
- polynomial, polynomial root;
- eigenvalue, eigenvector;

and be able to:

- perform elementary matrix operations (addition, multiplication, transposition, inversion);
- calculate a determinant and a trace of a matrix;
- find roots of a simple polynomial
- solve systems of linear equations.

II. Tasks

Task 1. Through direct calculations, compute eigenvalues and their corresponding eigenvectors for matrices:

a) $\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$;

b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$.

Task 2. The eigenvalues of the matrix A are eigenfrequencies of a bridge. μ is the resonant frequency of the structural driving force. The bridge can resonate if the distance between the frequency of the driving force and any of the eigenfrequencies is less than 1. Is it possible to prove (without direct calculations) using one of the known theorems that the bridge will not resonate?

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \ \mu = -5{,}5;$

b) $A = \begin{bmatrix} -1 & 1 & 3 \\ -2 & 11 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \ \mu = 7.$

Task 3. Is it possible to prove that all eigenvalues of the following matrix have negative real part by the Gerschgorin Circle Theorem?

$$A = \begin{bmatrix} -6 & 2 & -3 \\ -2 & -3 & 1 \\ 1 & 1 & -7 \end{bmatrix}.$$

Task 4. Using the power method, calculate the approximation of the dominant eigenvalue and its associated eigenvectors for the following matrices (carry out 4 iterations – stop at y^4):

a) $\begin{bmatrix} 4 & 1 & 3 \\ 5 & 1 & 4 \\ -1 & 1 & 0 \end{bmatrix}; y^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix}; y^0 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$

Task 5. Using the power method, list web pages by their Page Rank weightings assuming that the web consists of the following links: $1 \to 2, 2 \to 3, 3 \to 4,\ 4 \to 1, 1 \to 4, 2 \to 4$. Begin with the vector (1,1,1,1) and carry out 4 iterations.

Task 6. (additional) Calculate approximations of the rest of eigenvalues for the matrices from the task 4 a).

III. Homework

Task 1. Through direct calculations, compute eigenvalues and their corresponding eigenvectors for matrices:

a) $\begin{bmatrix} 1 & 4 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix};.$

b) $\begin{bmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{bmatrix}.$

Task 2. The eigenvalues of the matrix A are eigenfrequencies of a building. μ is the resonant frequency of the structural driving force. The bridge can resonate if the distance between the frequency of the driving force and any of the eigenfrequencies is less than 1. Is it possible to prove (without direct calculations) using one of the known theorems that the building will not resonate?

a) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 4 & 4 \\ 1 & 4 & 0 \end{bmatrix},\ \mu = 8;$

b) $A = \begin{bmatrix} 0 & 1 & 3 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ \mu = 7.$

Task 3. Is it possible to prove that all eigenvalues of the following matrices have negative real part by the Gerschgorin Circle Theorem?

a) $\begin{bmatrix} -3 & 2 & 3 \\ 1 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$;

b) $\begin{bmatrix} -2 & 1 & 0 \\ -2 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix}$.

Task 4. Using the power method, calculate the approximation of the dominant eigenvalue and its associated eigenvectors for the following matrices (carry out 4 iterations):

a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}; y^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

b) $\begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 2 & 1 & -4 \end{bmatrix}; y^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Task 5. Using the power method, list web pages by their Page Rank weightings assuming that the web consists of the following links:

a) $4 \to 1, 2 \to 3, 2 \to 4, 3 \to 4, \ 4 \to 2$;

b) $1 \to 2, \ 1 \to 4, \ 2 \to 1, \ 2 \to 3, 2 \to 5, 3 \to 1, 3 \to 2, 3 \to 4, 3 \to 5, 5 \to 2$.

Begin with the vector $(1, \dots, 1)^T$ and carry out 4 iterations.

Task 6. The population of rabbits currently consists of 24 rabbits younger than one year, 24 rabbits older than a year, but younger than two years and 20 rabbits older than two years. The maximum life span is 3 years. Half of the rabbits survive their first year. Of those, half survive their second year. Moreover during the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year. All the above data are represented by a vector x_0, and matrix A:

$$x_0 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

The population of rabbits in the next year is calculated by multiplying the current population of rabbits by a matrix A. Moreover the population is in equilibrium if in every year a population increase is proportional to the vector x, i.e. $Ax = \lambda x$, for some constant λ. Using the power method, performing six iterations, determine the dominant value of the constant λ and the corresponding vector. Answer the question: what age group in the population of rabbits will dominate, and what will be the least numerous?

Hint:
Note that λ is an eigenvalue of the matrix A and use power method to calculate it. In order to answer the last question in this task calculate the value of the eigenvector corresponding to eigenvalue λ.

IV. Answers

Task 1.

a) eigenvalues: $7,3,-1$, eigenvectors: $(2,3,2),(0,0,1),(-2,1,0)$;
b) eigenvalues: $11,7,0$, eigenvectors: $(1,2,0),(1,6,4),(-27,34,11)$;

Task 2.

a) yes, theorem 4.4 would work;
b) yes, Gerschgorin Circle theorem would work.

Task 3.

a) no;
b) yes.

Task 4.

a) $4{,}99$; $(1672{,}2500{,}5000)$; (actual dominant eigenvalue: 5)
b) $-2{,}55$; $(85{,}19{,}160)$ (actual dominant eigenvalue: $-2{,}81$).

Task 5.

a) $(4{,}1 = 2{,}3)$, eigenvector: $(0{,}972;\ 0{,}972;\ 0{,}683;\ 1{,}374)$

b) $(2{,}4{,}1 = 5{,}3)$, eigenvector: $(1{,}5;\ 2{,}749;\ 1{,}22;\ 2{,}555; 1{,}5)$.

Task 6.

Since $x_6 = \begin{bmatrix} 5256 \\ 1608 \\ 278 \end{bmatrix}$ therefore $\lambda_1 = \frac{6353047}{3352680} \approx 1{,}89$ (actual dominant eigenvalue: 2)

Eigenvector: $w^{(1)} = \begin{bmatrix} 3216 \\ 556 \\ 240 \end{bmatrix}$ (actual dominant eigenvector: $\begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix}$)

The group of rabbits younger than a year will dominate. The least numerous will be group of rabbits older than 2 years.