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Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy"
jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

# Numerical Methods

Handouts for students

## 3. Iterative methods for solving systems of linear equations

3.1. Iteration Method
3.2. Seidel Iteration Method
3.3. Convergence of iterative methods and stop condition

informatyka stosowana

przygotowane w ramach projektu „Uruchomienie
ȝo kierunku studiów Informatyka Stosowana odpowiedzią na
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UNIWERSYTET EKONOMICZNY W KRAKOWIE

## I. Introductory requirements

It is required to know the concepts of:

- vector, matrix, determinant of matrix;
- matrix equation;
- triagular matrix;

and be able to:

- make elementary operations on rows of matrix;
- multiply matrices;
- solve matrix equations.

## II. Classes

Task 1. What is the sufficient condition for the converegence of the iterative methods described by the schemes:

$$X^{(n+1)} = DX^{(n)} + C, \quad X^{(n+1)} = EX^{(n+1)} + FX^{(n)} + C.$$

Task 2. Check if the following matrices satisfy the sufficient condition for the converegence of the iterative methods:

$$A_1 = \begin{bmatrix} 0,7 & 0,3 & 0,2 \\ 0,4 & 0,9 & 0,3 \\ 0 & 0,5 & 0,7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0,1 & 0 & 1 \\ 0 & 0,9 & 1 \\ 0 & 0 & 0,3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 0 & -5 \\ 0 & 8 & 2 \\ -5 & 2 & 9 \end{bmatrix}$$

Task 3. Find the norm of the given matrices:

$$D_1 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \end{bmatrix}$$

and check how many iterations are needed to assure the error of estimation to be less then $10^{-2}$, if $X^{(0)} = [2 \ \ -1 \ \ 0]^T$.

Task 4. Find the first two approximations of the solution of the given systems in both methods, choosing $x^{(0)} = C$:

a) $$\begin{cases} 4x_1 - x_2 - x_3 = 0 \\ x_1 + 4x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 1 \end{cases}$$

b) $\begin{cases} 3x_1 - 1x_2 + x_3 = -1 \\ 2x_1 + 3x_2 \qquad = 0 \\ \qquad 2x_2 + 3x_3 = 0 \end{cases}$

Task 5. Write simplified Google matrix for the web:

$$1 \to 2, \quad 2 \to 3, \quad 2 \to 4, \quad 3 \to 1.$$

Using Iteration Method for corresponding system of linear equations evaluate wages and rank the pages.

Task 6. Reformulate the given system of linear equations to the system satisfying the sufficient condition for convergence of the iterative methods.

$$\begin{cases} 3x_1 + 2x_2 + 7x_3 = 0 \\ x_1 - x_2 + x_3 = 1 \\ 7x_1 + x_2 - 4x_3 = 2 \end{cases}$$

## III. Homework

Task 1. For the given systems of linear equations:

1. find the norm of the matrix $D$ in the iteration scheme and decide if the scheme is convergent or not;
2. find the minimal number of iterations necessary to receive the result with accuracy $\varepsilon = 10^{-9}$;
3. solve the system using one of the direct methods;
4. find the first two approximations of the solution in the Iteration Method;
5. find the first two approximations of the solution in the Seidel Iteration Method;
6. compare the results achieved in 3.,4. and 5.

a) $\begin{cases} 8x_1 + x_2 + x_3 = 4 \\ x_1 + 8x_2 + x_3 = 4 \\ x_1 + x_2 + 8x_3 = 4 \end{cases}$

b) $\begin{cases} -4x_1 - x_2 + \quad + x_4 = 4 \\ -2x_2 - x_3 = 1 \\ x_2 + 2x_3 = 4 \\ x_2 + 3x_3 - 8x_4 = -1 \end{cases}$

Task 2. We observe a rocket and note its velocity in three moments of time:

$$v(5) = 106{,}8 \text{ m/s}$$
$$v(8) = 177{,}2 \text{ m/s}$$
$$v(12) = 279{,}2 \text{ m/s}$$

The value of velocity is approximated by function $v(t) = a_1t^2 + a_2t + a_3$ dla $t \in [5{,}12]$. If it is possible, use the Seidel Method to find the value of coefficients $a_1, a_2, a_3$. Use vector $a^{(0)} = [1 \quad 2 \quad 5]^T$ as the first approximation and calculate the first two iterations.

Task 3. Write simplified Google matrix for the webs:

a) $1 \to 2,\ 2 \to 3, 3 \to 1,\ 3 \to 4, 4 \to 1, 4 \to 2;$

b) $1 \to 4,\ 1 \to 5, 1 \to 6,\ 2 \to 1,\ 2 \to 3, 3 \to 1, 3 \to 5,\ 5 \to 1,\ 5 \to 6,\ 6 \to 1,$ $6 \to 4$

Using the Iteration Method for appropriate system of linear equations estimate the value of weightings and rate the pages.

Task 4. Restaurant's kitchen is being equipped in three goods. Using the Seidel Iteration Method estimate the unit prices of those goods knowing that:
the price of two sets of those goods is 3,25 PLN;
two boxes of the first good and six boxes of the second one cost 5,05 PLN;
the cost of two boxes of the first good and five units of the third one is 3,5 PLN.

## IV. Answers

Task 1.

a)

1. $\|D\|_1 = \frac{1}{4}$
2. $k = 15$
3. $X = [0{,}4 \quad 0{,}4 \quad 0{,}4]^T$
4. $X^{(1)} = [0{,}375 \quad 0{,}375 \quad 0{,}375]^T$,
   $X^{(2)} = [0{,}40625 \quad 0{,}40625 \quad 0{,}40625]^T$;
5. $X^{(1)} = [0{,}375000 \quad 0{,}390625 \quad 0{,}404297]^T$,
   $X^{(2)} = [0{,}400635 \quad 0{,}399384 \quad 0{,}399998]^T$;

b)

1. $\|D\|_\infty = \frac{1}{2}$
2. $k = 31$
3. $X = [-0{,}25 \quad -2 \quad 3 \quad 1]^T$
4. $X^{(1)} = [-0{,}843750 \quad -1{,}5 \quad 2{,}25 \quad 0{,}8125]^T$,
   $X^{(2)} = [-0{,}421875 \quad -1{,}625 \quad 2{,}75 \quad 0{,}78125]^T$;
5. $X^{(1)} = [-0{,}84375 \quad -1{,}5 \quad 2{,}75 \quad 0{,}96875]^T$,
   $X^{(2)} = [-0{,}3828125 \quad -1{,}875 \quad 2{,}9375 \quad 0{,}9921875\ ]^T$.

Task 2. System of linear equations described in the text is following:

$$\begin{cases} 25a_1 + 5a_2 + a_3 = 106{,}8 \\ 64a_1 + 8a_2 + a_3 = 177{,}2 \\ 144a_1 + 12a_2 + a_3 = 279{,}2 \end{cases}$$

The main matrix of this system does not satisfy any of discussed during the lecture convergence conditions. Therefore no one could guarantee convergence of the iteration schemes in this case.

Task 3.Possible solutions:

a) For $w_4 = 2$, omitting the last equation we get:

$$\begin{cases} -w_1 + \frac{1}{2}w_3 = -1 \\ w_1 - w_2 = -1 \\ w_2 - w_3 = 0 \end{cases} \quad \begin{matrix} (W_1 + W_2) \\ (W_1 + W_2 + W_3) \\ \Leftrightarrow \end{matrix} \quad \begin{cases} -w_1 + \frac{1}{2}w_3 = -1 \\ -w_2 + \frac{1}{2}w_3 = -2 \\ -\frac{1}{2}w_3 = -2 \end{cases}$$

Ranking based on the direct solution: $(2 = 3,1,4)$
Ranking based on approximate solution: $(2 = 3,1,4)$
$k_{min} = 1$

b) For $w_4 = 6$, omitting the first equation we get the system satisfying convergence conditions:

$$\begin{cases} -w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3 + \frac{1}{2}w_5 + \frac{1}{2}w_6 = -1 \\ -w_2 = -1 \\ \frac{1}{2}w_2 - w_3 = -1 \\ \frac{1}{3}w_1 + \frac{1}{2}w_6 = 5 \\ \frac{1}{3}w_1 - w_5 = -1 \\ \frac{1}{3}w_1 + \frac{1}{2}w_3 + \frac{1}{2}w_5 - w_6 = -1 \end{cases} \quad \begin{matrix} \sout{(W_1)} \\ (W_1 \leftrightarrow W_6) \\ (W_1 + W_6 - W_4) \\ \Leftrightarrow \end{matrix} \quad \begin{cases} -\frac{5}{3}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_6 = -7 \\ -w_2 = -1 \\ \frac{1}{2}w_2 - w_3 = -1 \\ \frac{1}{3}w_1 - w_5 = -1 \\ \frac{1}{3}w_1 + \frac{1}{2}w_6 = 5 \end{cases}$$

Ranking based on the approximate solution: $(1,4 = 6,5,3,2)$
Ranking based on the direct solution: $(1,4,6,5,3,2)$
$k_{min} = 31$

Task 4. System of linear equations given in the task is following:

$$\begin{cases} 2p_1 + 2p_2 + 2p_3 = 3{,}25 \\ 2p_1 + 6p_2 = 5{,}05 \\ 2p_1 + 5p_3 = 3{,}5 \end{cases}$$

The main matrix of this system does satisfy the convergence condition. The unit prices of goods are, respectively: $p_1 = 0{,}3125, p_2 = 0{,}7375, p_3 = 0{,}5750$ (estimated by 70$^{th}$ iteration).