



**KAPITAŁ LUDZKI**  
NARODOWA STRATEGIA SPÓJNOŚCI



**UNIA EUROPEJSKA**  
EUROPEJSKI  
FUNDUSZ SPOŁECZNY



Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy”  
jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

---

# Numerical Methods

Handouts for students

## 2. Direct methods for solving systems of linear equations

- 2.1. Gauss Elimination Method
- 2.2. Gauss-Jordan Elimination
- 2.3. Google Page Rank Algorithm
- 2.4. LU Decomposition Method
- 2.5. Cholesky Method

## I. Introductory requirements

It is required to know the concepts of:

- vector, matrix, determinant of a matrix, rank of a matrix;
- inner product, norm of a vector, orthogonality of vectors;
- triangular matrix, diagonal matrix, orthogonal matrix;

and be able to:

- perform elementary matrix operations (addition, multiplication, transposition, inversion);
- calculate a determinant of a matrix and compute rank of a matrix;
- determine the number of solutions of a system of linear equations (Kronecker-Capelli theorem).

## II. Classes

Task 1. Using the Gauss Elimination Method solve following systems of equations:

$$a) \begin{cases} x_2 - x_3 + x_4 = 2 \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 = 1 \\ x_1 + 3x_2 - 2x_4 = 0 \\ x_2 - x_3 + 3x_4 = 3 \end{cases}$$

$$b) \begin{cases} -2x_1 + 5x_2 + 2x_3 - 2x_4 = -1 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ 3x_1 - 2x_2 + 2x_3 + x_4 = -3 \\ 2x_1 - 3x_2 + 3x_3 + 5x_4 = -7 \end{cases}$$

Task 2. Calculate the determinant of the matrix  $A$  using the Laplace expansion and using a computation schema of the Gauss Elimination.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -4 & -2 & 1 \\ -4 & 1 & 0 & -4 \\ 0 & 3 & 2 & 2 \end{bmatrix}$$

Task 3. Using the Gauss-Jordan Method solve following systems of equations:

$$a) \begin{cases} 2x_1 + x_3 - x_4 = 0 \\ -2x_2 + x_3 - x_4 = -8 \\ -x_1 + x_2 - 3x_3 + 4x_4 = 8 \\ 3x_1 - x_3 = 5 \end{cases}$$

$$b) \begin{cases} -x_2 + x_3 + x_4 = 8 \\ -x_1 - 3x_3 - x_4 = -15 \\ 2x_1 + x_3 - x_4 = 7 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = -3 \end{cases}$$

Task 4. Using the LU Decomposition Method solve following systems of equations:

$$a) \begin{cases} 2x_1 + 2x_2 - 4x_3 + 2x_4 = -2 \\ 2x_1 + 3x_2 - x_3 + x_4 = 0 \\ -2x_1 + 2x_2 + 15x_3 - 8x_4 = 5 \\ 5x_1 + 6x_2 - 7x_3 + 3x_4 = -5 \end{cases}$$

$$b) \begin{cases} -x_1 - 2x_2 + x_3 - 3x_4 = -1 \\ x_1 + 6x_2 - 5x_3 + 11x_4 = 1 \\ -2x_2 + 3x_3 - 5x_4 = 1 \\ -3x_1 - 8x_2 + 6x_3 - 12x_4 = 2 \end{cases}$$

Task 5. Using the Cholesky Method solve following systems of equations:

$$a) \begin{cases} x_1 + 4x_3 = 2 \\ x_2 + 2x_3 = -1 \\ 4x_1 + 2x_2 + 24x_3 = 6 \end{cases}$$

$$b) \begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ -x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - x_2 + 2x_3 - x_4 = -4 \\ x_1 - 2x_2 - x_3 + 7x_4 = 11 \end{cases}$$

Task 6. List web pages by their Page Rank weightings assuming that the web consists of the following links:

- a)  $1 \rightarrow 2, 2 \rightarrow 3, 2 \rightarrow 1;$
- b)  $4 \rightarrow 1, 2 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 4, 4 \rightarrow 2;$
- c)  $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 1, 1 \rightarrow 4, 2 \rightarrow 4, 2 \rightarrow 1;$
- d)  $1 \rightarrow 2, 1 \rightarrow 4, 2 \rightarrow 1, 2 \rightarrow 3, 2 \rightarrow 5, 3 \rightarrow 1, 3 \rightarrow 2, 3 \rightarrow 4, 3 \rightarrow 5, 5 \rightarrow 2;$
- e)  $1 \rightarrow 2, 2 \rightarrow 3, 2 \rightarrow 6, 3 \rightarrow 1, 3 \rightarrow 2, 3 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 6, 6 \rightarrow 5, 6 \rightarrow 1;$

### III. Homework

Task 1. Using the Gauss Elimination Method solve following systems of equations:

$$a) \begin{cases} 6x_1 + 3x_2 + 4x_3 + 3x_4 = 3 \\ 2x_1 - 3x_2 + 6x_3 + 2x_4 = 4 \\ 4x_1 + 4x_3 - 2x_4 = 2 \\ -2x_1 + 3x_2 + x_3 + x_4 = -3 \end{cases}$$

$$b) \begin{cases} 2x_1 + x_2 + 4x_3 + 2x_4 = 2 \\ -3x_1 + 2x_2 + 2x_3 + 3x_4 = 1 \\ -x_1 - 4x_2 - 2x_3 + x_4 = -5 \\ 4x_1 + x_2 + 5x_3 - 5x_4 = -7 \end{cases}$$

Task 2. Calculate determinants of matrices  $A$  and  $B$  using the Laplace's formula and using a computation schema of the Gauss Elimination Method.

$$A = \begin{bmatrix} -2 & 3 & 1 & 5 \\ 1 & -2 & 0 & -2 \\ -3 & 1 & 3 & 3 \\ -4 & 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & 2 & 1 & 2 \\ -1 & -2 & 3 & 2 & -1 \\ 2 & 2 & 0 & -1 & -2 \\ 1 & -1 & 4 & 0 & 1 \\ 0 & 1 & 2 & -1 & -1 \end{bmatrix}$$

Task 3. Using the Gauss-Jordan Method find solutions of systems of equations:

$$a) \begin{cases} -3x_1 + x_2 + 2x_3 + 4x_4 = 2 \\ 2x_1 - 2x_2 + 3x_3 + 2x_4 = 4 \\ -x_2 + x_3 - x_4 = -5 \\ -2x_1 + 3x_2 - 2x_3 + 2x_4 = 1 \end{cases}$$

$$b) \begin{cases} 2x_1 + 7x_3 + 2x_4 = 5 \\ -x_1 - x_2 + x_3 + x_4 = 3 \\ 4x_1 - 3x_2 - x_3 - x_4 = 2 \\ 3x_1 + 2x_2 + 3x_3 - 2x_4 = -3 \end{cases}$$

Task 4. Solve following systems of equations using the LU Decomposition Method:

$$a) \begin{cases} 3x_1 - 3x_2 + 6x_3 + 3x_4 = 6 \\ x_1 + x_2 + 2x_3 - 3x_4 = 6 \\ -x_1 + 3x_2 - x_3 - 2x_4 = 1 \\ x_1 + x_2 + 5x_3 + 7x_4 = 2 \end{cases}$$

$$b) \begin{cases} 2x_1 - 2x_2 + 2x_3 + 2x_4 = 8 \\ x_1 - 2x_2 + 4x_3 + 3x_4 = 9 \\ -x_1 - x_2 + 6x_3 + 6x_4 = 12 \\ x_1 + 2x_2 - 5x_3 + 5x_4 = 9 \end{cases}$$

Task 5. Using the Cholesky Method solve following systems of equations:

$$\begin{aligned} \text{a)} \quad & \begin{cases} 9x_1 - 3x_2 + 6x_3 = 3 \\ -3x_1 + 5x_2 = 3 \\ 6x_1 + 6x_3 = 6 \end{cases} \\ \text{b)} \quad & \begin{cases} x_1 - x_2 - 2x_3 + x_4 = 0 \\ -x_1 + 5x_2 + 4x_3 - 3x_4 = 10 \\ -2x_1 + 4x_2 + 6x_3 - x_4 = 4 \\ x_1 - 3x_2 - x_3 + 7x_4 = -7 \end{cases} \end{aligned}$$

Task 6. Solve systems of equations by List web pages by their Page Rank weightings assuming that the web consists of the following links:

- a)  $1 \rightarrow 2, 2 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1;$
- b)  $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 1, 4 \rightarrow 2;$
- c)  $1 \rightarrow 4, 1 \rightarrow 5, 1 \rightarrow 6, 2 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 1, 3 \rightarrow 5, 5 \rightarrow 1, 5 \rightarrow 6, 6 \rightarrow 1, 6 \rightarrow 4;$

#### IV. Answers

Task 1.

$$\begin{aligned} \text{a)} \quad & X = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}^T; \\ \text{b)} \quad & X = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}^T. \end{aligned}$$

Task 2.

$$\det A = -6, \det B = 12.$$

Task 3.

$$\begin{aligned} \text{a)} \quad & X = [-3 \quad -15 \quad -12 \quad 8]^T; \\ \text{b)} \quad & X = \begin{bmatrix} -\frac{1}{3} & -\frac{3}{2} & \frac{2}{3} & \frac{1}{2} \end{bmatrix}^T. \end{aligned}$$

Task 4.

$$\begin{aligned} \text{a)} \quad & X = [-1 \quad 0 \quad 2 \quad -1]^T; \\ \text{b)} \quad & X = [1 \quad -1 \quad 0 \quad 2]^T. \end{aligned}$$

Task 5.

$$\begin{aligned} \text{a)} \quad & X = [-1 \quad 0 \quad 2]^T; \\ \text{b)} \quad & X = [1 \quad 3 \quad -1 \quad 0]^T. \end{aligned}$$

Task 6.

- a)  $(2,1,3 = 4);$
- b)  $(2 = 3,1,4);$
- c)  $(1,4,6,5,3,2).$