Projekt „Uruchomienie unikatowego kierunku studiów Informatyka Stosowana odpowiedzią na zapotrzebowanie rynku pracy”
jest współfinansowany ze środków Unii Europejskiej w ramach Europejskiego Funduszu Społecznego.

Numerical Methods

Classes 1-2

Handouts for students

1. Error theory, big *O* notation

1.1. Absolute error, relative error
1.2. General error formula
1.3. Inverse problem in the error theory
- equal influence rule
- equal upper bounds for the absolute errors rule
- equal measurement rule

1.4. Big *O* notation

przygotowane w ramach projektu „Uruchomienie
go kierunku studiów Informatyka Stosowana odpowiedzią na
wanie rynku pracy” ze środków Programu Operacyjnego
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I. Introductory requirements

It is required to know the concepts of:

- derivative of function;
- partial derivative of multiple variable function;
- differential of function;
- Taylor series for multiple variable function;

and be able to:

- calculate derivatives;
- calculate partial derivatives;
- solve logarythmic inequalities.

If the task does not specify precision then calculations should be made with precision of (at least) four decimal places.

II. Classes

Task 1. Calculate an absolute error Δa and a relative error δa, if a number A is approximated by its truncation to four decimal places. What are the upper bounds for the absolute error Δ_a and the relative error δ_a (with precision to five decimal places)?

a) $A = 0{,}321947$,

b) $A = e \approx 2{,}7182818$,

Task 2. The hard drive in a computer has capacity of 120000 MB. The user of this computer has estimated that he can save about 120 GB of data on this disc. Knowing that 1 GB = 1024 MB, calculate the real disk capacity in GB and then calculate the absolute error and relative error for estimation of disk capacity in GB.

Task 3. Calculate the upper bounds of absolute error and the relative error of a value of marginal aggregate demand with estimated prices of goods: $p_1 = 1{,}99$, $p_2 = 3{,}11$, $p_3 = 2{,}10$ if the demand is given by the formula:

$$Q(p_1, p_2, p_3) = \frac{2}{p_1} + \frac{3}{p_2} + \frac{4}{p_3}.$$

Task 4. The volume of one kilogram of a gas is given by the formula

$$V(x, y, z) = x^2 z + \sqrt{\frac{x}{y}},$$

where x, y, z are numbers of moles of its three components. Calculate the upper bounds of absolute and relative errors of the volume of one kilogram of the gas, if the number of molecules of its components are $x = 4 \pm 0{,}04$, $y = 1 \pm 0{,}02$, $z = 2 \pm 0{,}01$.

Task 5. The area of a triangular parcel can be calculated using the Heron's formula

$$P(a,b,c) = \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2},$$

where a, b, c denote lengths of the edges of the parcel. What are the admissible values of absolute errors of the values $a \approx 3m$, $b \approx 4m$, $c \approx 5m$ such that the absolute error of the area of the parcel does not exceed 0,1? Solve this task using three methods: equal influence rule, equal upper bounds rule and equal measurement rule.

Task 6. The radius of the base of a cylinder is $r \approx 2cm$, and the height of the cylinder is $h \approx 3cm$. Calculate admissible values of absolute errors of r, h and π such that the absolute error of the volume of this cylinder does not exceed $0{,}1cm^3$ (Assume that $\pi \approx 3{,}14$).

Task 7. Due to the forthcoming end of the support for the operating system Windows XXL an administrator of the university network plans to update the software on the university's machines. For this purpose he prepares a list of n operating systems, and considers implementation of a sorting algorithm that could be used to sort the list with respect to various criteria. The administrator is considering the following algorithms:

a) Insertion sort: in the k −th step $(1 \leq k \leq n)$ the first $k-1$ elements are already sorted. Choose the k −th element of the list and inset it on the appropriate position by comparing it with the elements of the sorted part of the list.
b) Selection sort: in the k −th step $(1 \leq k \leq n)$ the $k-1$ smallest elements are already sorted. Select the smallest element of the unsorted part of the list and replace it with the element on the k −th position.

Determine the computational complexity of those algorithms (by calculating the number of comparison operations) and select the better one (with respect to complexity).

Task 8. Compute the number of multiplications and the number of additions performed during solving a system of n linear equations with n variables by using the Gauss-Jordan elimination, and determine the computational complexity of the algorithm.

III. Homework

Task 1. Calculate an absolute error Δa and a relative error δa, if a number A is approximated by a. What are the upper bounds for the absolute error Δ_a and the relative error δ_a?

a) $A = 1{,}129124$, $a = 1{,}12$;
b) $A = 1{,}129124$, $a = 1{,}13$;
c) $A = \pi$, $a = 3{,}14$

Task 2. Given a function f and estimated values of arguments calculate an absolute error and a relative error of a value of the function:

a) $f(x,y) = \log x + \log y,\ x = 10 \pm 0{,}001,\ y = \frac{1}{10} \pm 0{,}0001;$
b) $f(x,y,z) = (x+y)^2 + (y+z)^2 - (x+z)^2,\quad x = 1 \pm 0{,}02,\quad y = -1 \pm 0{,}03,\ z = 3 \pm 0{,}02.$

Task 3. According to the Newton's law of universal gravitation two bodies attract each other with a force

$$F(M,m,r) = \frac{GMm}{r^2},$$

where M, m are the masses of those bodies, r is the distance between the centers of the masses and G is the gravitational constant (we assume that $G = \frac{20}{3}\frac{m^3}{kg\, s^2}$). Calculate the absolute error and the relative error of the value of gravitational force between bodies if their masses are $M = 4 \pm \frac{9}{100} kg$ and $m = 2 \pm \frac{3}{100} kg$ respectively and the distance between the centers of the masses is $r = 2 \pm \frac{12}{1000}$ meters.

Task 4. Given a function f and estimated values of arguments x, y and z calculate admissible values of absolute errors of arguments such that the absolute error of the value of f does not exceed $0{,}03$. Calculate the result using three different methods and compare the results.

a) $f(x,y,z) = y - xyz,\quad x \approx 2,\ y \approx 5,\ z \approx 1;$
b) $f(x,y,z) = e^{x+y+z}(x^2 + y^2 + z^2),\quad x \approx 1,\ y \approx 0,\ z \approx -1.$

Task 5. The space station *Miś*, placed in the Earth orbit, received a message with an order to correct its position. For the change of the position the station uses three engines. The engines should work $x \approx 2$ minutes, $y \approx 4$ minutes and $z \approx 3$ minutes respectively. The change in the position of the station (in meters) is described by a function

$$f(x,y,z) = xy + yz - xz.$$

Calculate the admissible values of absolute errors of x, y and z, so that the error of the position of the station does not exceed 0,3 meters. Calculate the result using three different methods and write it in seconds.

Task 6. Determine the number of multiplications and the number of additions performed during:

a) computation of the inverse of a $n \times n$ matrix by using the Gauss-Jordan elimination;
b) computation of the rank of a $n \times n$ matrix by using the Gaussian elimination.
Determine the computational complexity of those algorithms.

IV. Answers

Task 1.

a) $\Delta a = 0{,}009124$, $\Delta_a = 0{,}0092$
$\delta a \approx 0{,}00808$, $\delta_a = 0{,}0081$

b) $\Delta \mathrm{a} = 0{,}000876$, $\Delta_\mathrm{a} = 0{,}0009$
$\delta \mathrm{a} \approx 0{,}000776$, $\delta_\mathrm{a} = 0{,}0008$

c) $\Delta \mathrm{a} = \pi - 3{,}14 \approx 0{,}0015926535$, $\Delta_\mathrm{a} = 0{,}0016$
$\delta \mathrm{a} = \frac{\pi - 3{,}14}{\pi} \approx 0{,}000506957$, $\delta_\mathrm{a} = 0{,}0006$

Task 2.

a) $|\Delta f| = 0{,}0011$, $f(x,y) = \pm 0{,}0011$, δf – may not be applied
b) $|\Delta f| = 0{,}36$, $f(x,y,z) = -12 \pm 0{,}36$, $\delta_f = 0{,}03$.

Task 3.

$$\Delta F = \frac{66}{100}\frac{kg\ m}{s^2}, \delta F = \frac{99}{2000} = 4{,}95\%$$

Task 4.

equal influence rule:

a) $\Delta_x = 0{,}002$, $\Delta_y = 0{,}01$, $\Delta_z = 0{,}001$;
b) may not be applied ($f_z'(1{,}0,-1) = 0$).

equal upper bounds rule:

a) $\Delta_x = \Delta_y = \Delta_z = 0{,}001875$;
b) $\Delta_x = \Delta_y = \Delta_z = 0{,}005$.

equal measurement rule:

a) $\Delta_x = 0{,}0024$, $\Delta_y = 0{,}006$, $\Delta_z = 0{,}0012$;
b) $\Delta_x = 0{,}0075$, $\Delta_y = 0$, $\Delta_z = 0{,}0075$.

Task 5.

equal influence rule: $\Delta_x = 6$ seconds, $\Delta_y = \frac{6}{5}$ seconds, $\Delta_z = 3$ seconds equal upper bounds rule: $\Delta_x = \Delta_y = \Delta_z = \frac{9}{4}$ seconds

equal measurement rule: $\Delta_x = \frac{9}{7}$ seconds, $\Delta_y = \frac{18}{7}$ seconds, $\Delta_z = \frac{27}{14}$ seconds

Task 6.

a) number of multiplications:

$$\sum_{k=1}^{n}[n-(k-1)+(n-(k-1))(n-1)]+\sum_{k=1}^{n}[n+n(n-1)]=\frac{1}{2}n^2(3n+1),$$

number of additions:

$$\sum_{k=1}^{n}(n-(k-1))(n-1)+\sum_{k=1}^{n}n(n-1)=\frac{1}{2}(n-1)n(3n+1),$$

total number of operations: $\frac{1}{2}\mathrm{n}(2\mathrm{n}-1)(3\mathrm{n}+1)$,

computational complexity: $\mathrm{O}(\mathrm{n}^3)$.

b) number of multiplications:

$$\sum_{k=1}^{n}\big(n-(k-1)\big)(n-1-(k-1))=\frac{1}{3}(n-1)n(n+1),$$

number of additions:

$$\sum_{k=1}^{n}\big(n-(k-1)\big)(n-1-(k-1))=\frac{1}{3}(n-1)n(n+1),$$

total number of operations: $\frac{2}{3}(\mathrm{n}-1)\mathrm{n}(\mathrm{n}+1)$,

computational complexity: $\mathrm{O}(\mathrm{n}^3)$.